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Twin-characteristic-parameter solution for dynamic buckling of columns under elastic compression wave

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Abstract

Two critical conditions for the dynamic buckling of columns are derived on the basis of the consideration of energy transformation and conservation at the instant when the buckling occurs. The first critical condition is that the amount of released compressive deformation energy must be equal to the sum of buckling deformation energy and buckling kinetic energy in the instant course of the dynamic buckling. The second is that the rate of energy transformation meets the conservation law in the instant course. The governing equations, the boundary conditions and the continuity conditions derived by use of the first condition are the same as those obtained by use of Hamilton's theorem. These equations and conditions are insufficient for the determination of the critical load parameter and the exponent of transverse inertia term involved in the problem. A supplementary restraint equation at compression wave front is derived by use of the second condition.

Two characteristic equations for the two parameters are derived by use of the solution of the governing equations and the above-mentioned restraint conditions. The two characteristic parameters and the corresponding buckling modes are calculated accurately from the solution of the characteristic equations. The simple formulas for the relation of the critical force with the buckling time are given. The theoretical results predicted by use of the formula are in reasonable agreement with the existent experiment results. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Dynamic buckling; Instability criterion; Elastic column; Characteristic parameter solution; Stress wave

1. Introduction

The buckling problem of columns under an axial load applied dynamically has been studied by many investigators (Koning and Taub, 1933; Meier, 1945; Gerard and Becker, 1952; Davidson, 1953; Hoff, 1953; Sevin, 1960; Hutchinson and Budiansky, 1966; Lindberg, 1966; Hayashi and Sano, 1972a,b; Ari-Gur et al., 1982; Housner and Knight, 1983; Lindberg and Florence, 1983). A review for these studies was given by Simitses (1987). In most of these studies, the column under consideration was assumed to have an initial imperfection. When some characteristic deflection increases rapidly with time, a dynamically critical

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condition is obtained. The buckling deformation mode and the corresponding critical load determined in this way may be influenced by the initial imperfection. In addition to the above-mentioned studies, Lee (1981) presented a quasi-bifurcation method for the dynamic buckling analysis of inelastic columns. Kounadis and Raftoyiannis (1990) presented a nonlinear dynamic buckling analysis for discrete systems with a limit point or an unstable branching point under static load. Tang and Zhu (1994) investigated the relation between critical time and axial thrust corresponding to the first mode for the column clamped at the impact end, and the experimental verification for the theoretical prediction was made.

Experimental results (Hayashi and Sano, 1972a,b; Lindberg and Florence, 1983; Tang and Zhu, 1994) show that at an early stage of the impact the local buckling occurs near the impact end when the impact velocity is high. From the experimental results, it may be concluded that it is necessary to consider the propagation effects of the axial compression wave in the investigation.

An important problem in the investigation for the dynamic buckling of columns is how to determine quantitatively the transverse inertia effect. This is essential to the accurate determination of the critical load and the corresponding buckling mode. The investigation results of this paper will show that the transverse inertia effect plays an important role in the dynamic buckling of columns.

In studies on the instability of structures under static loads, the buckling mode and the corresponding critical load can be calculated first for the structure without imperfection, then the influence of imperfection on the buckling is investigated. Following the procedure of the static buckling analysis for a perfect column, we may present the method of the dynamic buckling analysis of the perfect column. In the linear analysis of the static buckling, an algebraic eigenvalue problem is deduced in accordance with the condition under which the stability equation of the column has a nontrivial solution satisfying the boundary conditions. By solving the eigenvalue problem, the buckling mode and the critical load of the column are calculated accurately. In the static buckling problem, only one characteristic parameter must be determined, which is the critical load parameter. In contrast to the static buckling problem, there are two undetermined characteristic parameters in the dynamic buckling problem, which are the critical load parameter and the exponent parameter related to the transverse inertia term. The exponent parameter is named the dynamic characteristic parameter in this paper.

For the dynamic buckling of columns, governing equations, boundary conditions and continuity conditions at compression wave front can be derived by use of the adjacent-equilibrium criterion (Brush and Almroth, 1975) and Hamilton's principle. Only one characteristic equation can be derived from the condition under which the governing equations have a nontrivial solution satisfying the boundary conditions and the continuity conditions. It is obvious that one characteristic equation is insufficient for determining the two characteristic parameters as above mentioned.

In this paper, in order to obtain the sufficient conditions of determining the two parameters, two critical conditions for the dynamic buckling of columns are derived on the basis of the law of energy transformation and conservation at the instant when the dynamic occurs. The first critical condition is that the amount of the released compressive deformation energy must be equal to the sum of the buckling deformation energy and the buckling kinetic energy. The second critical condition is that the rate of energy transformation meets the energy conservation law in the instant course of the buckling. The governing equations, the boundary conditions and the continuity conditions derived by use of the first critical condition are the same as those obtained by use of Hamilton's theorem. A supplementary restraint equation at the front of the compression wave is derived by use of the second critical condition. A couple of characteristic equations for the two characteristic parameters are derived in accordance with the condition under which the governing equations have a nontrivial solution satisfying the boundary conditions, the continuity conditions and the supplementary restraint equation. The dynamic buckling modes, the critical load parameter and the dynamic characteristic parameter are calculated accurately from the solutions of the characteristic equations.

2. Compression wave and axial force in column

As shown in Fig. 1a, we consider a straight uniform column of length L and cross-section area A. The section inertia moment of the column is I. The Young's modulus of the column material is E and the material density is ρ. The loaded end of the column is movable in the axial direction and the other end is fixed. At the instant $t_0 = 0$, an axial step load with the magnitude N is applied suddenly at the loaded end A, at the same time an elastic compression wave starts propagating toward the fixed end B from the loaded end, at the velocity $c = \sqrt{E/\rho}$. In this paper, we deal with the dynamic buckling that occurs at the first two stages of the compression wave propagation. In order to avoid the discussion of unloading problem, we assume that the loading duration is larger than $2L/c$.

(a) The first stage is defined as $0 \leqslant t \leqslant L/c$. At this stage, the compression wave front is propagating from the loaded end toward the fixed end, and the compression wave has not reflected from the fixed end. The characteristic representing the position of the wave front and the axial force in the column are illustrated in Fig. 1a. At any instant t of this stage, the propagation distance of the wave front from the loaded end is

$$L_1 = ct \tag{2.1}$$

(b) The second stage is defined as $L/c \leqslant t \leqslant 2L/c$. For this stage, the right end B of the column is designated as the loaded end and the origin of the x-axis is located at the left end A, as shown in Fig. 1b. At this stage, the compression wave has already arrived at the fixed end and reflected from the fixed end for the first time, and the reflected wave front is traveling toward the loaded end. For the convenience of analysis, the instant when the wave front arrives at the fixed end is written as $t_0 = 0$. At any instant t of this stage, the reflected wave front is at a distance $L_1 = ct$ from the reflection end, as shown in Fig. 1b. In the region $0 < x < L_1$, the axial stress caused by the initial compression wave together with the reflected wave is obtained by superposition (Timoshenko and Goodier, 1970). In this paper, we only consider the case where the reflection end is fixed in the axial direction and the axial stress in the region $0 < x < L_1$ does not exceed the material yield stress. Under the above-mentioned conditions, the axial force in the region $0 < x < L_1$ is twice as much as that in the region $L_1 < x < L$, as shown in Fig. 1b.

As shown by the experiments (Hayashi and Sano, 1972a,b; Tang and Zhu, 1994), the dynamic buckling will occur at the first stage if the amplitude N of the applied load is large enough. The dynamic buckling

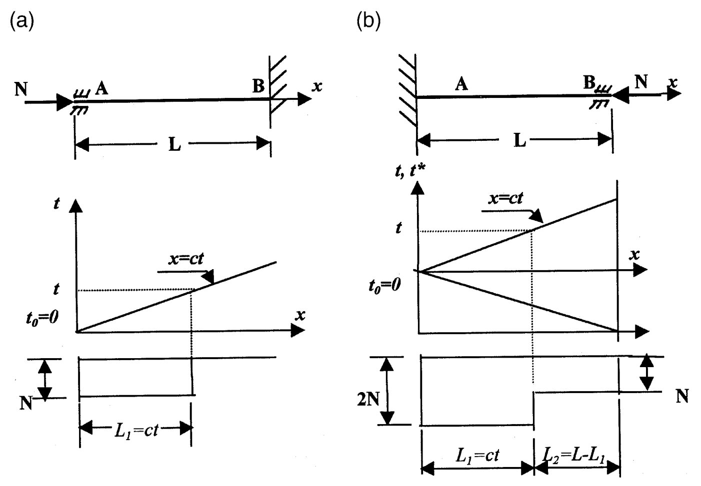

Fig. 1. (a) Axial force in column before compression wave is reflected from the fixed end. (b) Axial force in column after compression wave is reflected for the first time.

may also take place at the second stage or after the second stage if the load amplitude is lower and the loading duration is long enough. As mentioned above, the investigation of this paper will be confined to the dynamic buckling occurring at the first and the second stages of the compression wave propagation.

3. Governing equations for dynamic buckling at the first stage

3.1. Governing equations and boundary conditions derived by Hamilton's theorem

Before the dynamic buckling occurs, the column maintains the straight configuration, as shown in Fig. 1a. For the portion $0 \leqslant x < ct$, the axial displacement of the unbuckled column is denoted by u_0. The axial strain ε_0, the axial velocity $\dot{u}_0$, and the axial force N_{x0} are given respectively by the following expressions:

$$\varepsilon_0 = u_{0,x}, \quad \dot{u}_0 = \frac{du_0}{dt} = \frac{Nc}{EA}, \quad N_{x0} = -N \qquad (0 \leqslant x < ct) \tag{3.1}$$

Let u_1 and w denote respectively the infinitesimal axial-displacement increment and the infinitesimal deflection of the portion $0 \leqslant x < ct$ of the column at the initial instant when the dynamic buckling occurs. The total axial displacement and the total axial strain at the point on the centroidal line of the column are given respectively by following equations:

$$u = u_0 + u_1, \quad \varepsilon = \varepsilon_0 + \varepsilon_1, \quad \varepsilon_1 = u_{1,x} + \frac{1}{2}w_{,x}^2 \qquad (0 \leqslant x < ct) \tag{3.2}$$

The displacements $(u_0, w_0 = 0)$ and (u, w) represent the two adjacent configurations corresponding to the same applied load. The portion $ct < x \leqslant L$ of the column remains undisturbed, where the compression wave front has not reached at the instant when the dynamic buckling occurs.

Considering that the effects of rotational inertia and shear deformation are small (Hayashi and Sano, 1972b), we may omit these effects and write the expressions of deformation energy U and kinetic energy K for the buckled column, as follows:

$$U = U_1 + U_2 = \frac{1}{2}\int_0^{ct} N_x\varepsilon_x \, dx + \frac{1}{2}EI\int_0^{ct} (w_{,xx})^2 \, dx \tag{3.3}$$

$$K = K_1 + K_2 = \frac{1}{2}\rho A\int_0^{ct} \dot{w}^2 \, dx + \frac{1}{2}\rho A\int_0^{ct} \dot{u}^2 \, dx \tag{3.4}$$

The work done by the external force is

$$W = Nu(0, t) \tag{3.5}$$

Introducing Eqs. (3.3)–(3.5) into the formula of Hamilton's theorem (Fung, 1965)

$$\delta\int_{t_1}^{t_2} (U - K - W) \, dx = 0 \tag{3.6}$$

From Eq. (3.6), we obtain the following governing equations and the boundary conditions

$$E(u_{1,xx} + w_{,x}w_{,xx}) = \rho\ddot{u} \tag{3.7}$$

$$EIw_{,xxxx} + (Nw_{,x})_{,x} + \rho A\ddot{w} = 0 \tag{3.8}$$

$$\left(u_{1,x} + \frac{1}{2}w_{,x}^2\right)_{x=0} = 0, \quad u_1(L_1, t) = 0 \tag{3.9}$$

$$\begin{aligned} &w(0,t)=0, \quad \text{or} \quad (Nw_{,x}+EIw_{,xxx})_{x=0}=0 \\ &\{w_{,x}\}_{x=0}=0, \quad \text{or} \quad \{w_{,xx}\}_{x=0}=0 \end{aligned} \tag{3.10}$$

$$w(L_1,t)=0, \quad \{w_{,x}\}_{x=L_1}=0 \tag{3.11}$$

In Eqs. (3.9)–(3.11), the coordinates $x=0$ and $x=L_1=ct$ correspond to the loaded end and the compression wave front, respectively.

By separation of variables, the deflection function $w(x,t)$ is written as

$$w(x,t)=Y(x)T(t), \quad u(x,t)=Z(x)T_1(t) \tag{3.12}$$

Introducing the first term of Eq. (3.12) into Eq. (3.8) gives

$$\ddot{T}-\lambda\gamma^2T=0, \quad Y''''(x)+\alpha^2Y''(x)+\lambda Y(x)=0 \tag{3.13}$$

where dotes and primes denote respectively differentiation with respect to the time variable t and the axial coordinate x, and λ is the undetermined parameter that is named the dynamic characteristic parameter. The parameters α and γ are respectively defined as

$$\alpha^2=\frac{N}{EI}, \quad \gamma^2=\frac{EI}{\rho A} \tag{3.14}$$

For the dynamic buckling, the deflection w increases with the time variable t. Therefore, the solution of the first term of Eq. (3.13) is taken as

$$\lambda=\omega^2>0, \quad T=C_0\mathrm{e}^{\gamma\omega(t-\tau)} \tag{3.15}$$

where τ denotes the critical time when the dynamic instability occurs, and C_0 is an infinitesimal integration constant. From Eqs. (3.7), (3.12) and (3.15), we obtain

$$T_1=T^2=C_0^2\mathrm{e}^{2\gamma\omega(t-\tau)}, \quad Z''(x)+Y'(x)Y''(x)=\frac{4\omega^2I}{A}Z(x) \tag{3.16}$$

By use of Eqs. (3.12), the boundary conditions Eqs. (3.9)–(3.11) are transformed into

$$Z'(0)+\frac{1}{2}\left[Y'(0)\right]^2=0, \quad Z(ct)=0 \tag{3.17}$$

$$\begin{aligned} &Y(0)=0 \quad \text{or} \quad \alpha^2Y'(0)+Y'''(0)=0 \\ &Y'(0)=0 \quad \text{or} \quad Y''(0)=0 \end{aligned} \tag{3.18}$$

$$Y(ct)=0, \quad \text{and} \quad Y'(ct)=0 \tag{3.19}$$

The present dynamic instability analysis is to solve the second term of Eq. (3.13) under the boundary conditions (3.18) and (3.19). In this problem, there are the two undetermined characteristic parameters, which are the critical load parameter α^2 and the dynamic characteristic parameter $\lambda=\omega^2$. Only one characteristic equation is derived from the condition under which the second term of Eq. (3.13) has a nontrivial solution meeting the boundary conditions (3.18) and (3.19). It is insufficient for the determination of the two characteristic parameters. In order to obtain the sufficient conditions of determining the two parameters, it is necessary to present the new criterion for the dynamic instability of the column.

3.2. The criterion of energy transformation and conservation and the supplementary restraint equation at the compression wave front

Let us check the variation of the deformation energy and the kinetic energy of the column in the instant course of the dynamic buckling. Neglecting the shear deformation and considering that u_1, ε_1 and N_{x1} are arbitrarily small increments in contrast with the quantities u_0, ε_0 and N_{x0}, we write the increment of the deformation energy of the column as

$$\Delta U = -\int_0^{ct} N(u_{1,x} + \frac{1}{2}w_{,x}^2)\,dx + \frac{1}{2}EI\int_0^{ct} w_{,xx}^2\,dx \tag{3.20}$$

With the rotational inertia effect omitted, the increment of the kinetic energy is written as

$$\Delta K = K_1 + \Delta K_2 = \frac{1}{2}\rho A\int_0^{ct} \dot{w}^2\,dx + \frac{1}{2}\rho A\int_0^{ct}(2\dot{u}_0\dot{u}_1 + \dot{u}_1^2)\,dx \tag{3.21}$$

From Eqs. (3.1), the second term of Eqs. (3.12), (3.16) and (3.17), it is proved that

$$\int_0^{ct}(\dot{u}_0\dot{u}_1)\,dx = 0, \quad \Delta K_2 = \frac{1}{2}\rho A\int_0^{ct}\dot{u}_1^2\,dx \tag{3.22}$$

By use of Eqs. (3.7) and (3.16), we can prove that $\dot{u}_1$ is an infinitesimal with the same order of magnitude as $\dot{w}^2$. Therefore, ΔK_2 may be omitted from Eq. (3.21), and we obtain

$$\Delta K = K_1 = \frac{1}{2}\rho A\int_0^{ct}\dot{w}^2\,dx \tag{3.23}$$

In the instant process of the dynamic buckling, the increment of the work done by the external load is

$$\Delta W = Nu_1(0,t) \tag{3.24}$$

According to the law of energy conversation, there must be

$$\Delta W = \Delta U + \Delta K \tag{3.25}$$

We write

$$U_{\text{pre}} = \frac{1}{2}\int_0^{ct} Nw_{,x}^2\,dx \tag{3.26}$$

$$U_{\text{buc}} = \frac{1}{2}EI\int_0^{ct} w_{,xx}^2\,dx \tag{3.27}$$

From Eqs. (3.20) and (3.23)–(3.27), the following equation is obtained.

$$U_{\text{pre}} = U_{\text{buc}} + \Delta K \tag{3.28}$$

In Eqs. (3.26), (3.27) and (3.23), U_{pre} denotes the compressive deformation energy released owing to the buckling, U_{buc} represents the buckling deformation energy and ΔK is the buckling kinetic energy. In fact, the buckling of a structure at the bifurcation point is an instant process of releasing partially compressive deformation energy (Evans and Hutchinson, 1984; Wang, 1999). When the buckling takes place, the structure abruptly gets into an adjacent configuration from its initial configuration. This instant process is accompanied with the release of the partial compressive deformation energy. The released compressive deformation energy transforms into the buckling deformation energy and the buckling kinetic energy of the structure. Eq. (3.28) is the critical condition of the dynamic buckling of the column. It shows that amount of the released compression energy must be equal to the sum of the buckling deformation energy and the buckling kinetic energy in the instant course of the dynamic buckling.

Differentiating both sides of Eq. (3.28) with respect to the time variable t, we obtain the second critical condition for the dynamic buckling of the column

$$\dot{U}_{\rm pre} = \dot{U}_{\rm buc} + \Delta\dot{K} \tag{3.29}$$

Eq. (3.29) shows that the energy transformation rate meets the conservation law in the buckling process. The critical conditions (3.28) and (3.29) compose the criterion of the dynamic instability of the column, which may be named the criterion of energy transformation and conservation. For the static bifurcation instability, $K = 0$, only critical condition (3.28) is needed.

Substituting Eqs. (3.23), (3.26) and (3.27) into the critical condition (3.28), by use of first term of Eq. (3.12), we re-obtain the first term of Eq. (3.13) and derive the following equation.

$$\alpha^2 \int_0^{ct} \left[Y'(x)\right]^2 \mathrm{d}x - \int_0^{ct} \left[Y''(x)\right]^2 \mathrm{d}x - \lambda \int_0^{ct} [Y(x)]^2 \mathrm{d}x = 0 \tag{3.30}$$

Integrating Eq. (3.30) by parts, we re-obtain the second term of Eq. (3.13) and the boundary conditions (3.18) and (3.19).

Introducing Eqs. (3.23), (3.26) and (3.27) into the critical condition (3.29), we derive the following equation.

$$\begin{aligned} &EIT\dot{T}\left\{\alpha^2 \int_0^{ct} \left[Y'(x)\right]^2 \mathrm{d}x - \int_0^{ct} \left[Y''(x)\right]^2 \mathrm{d}x - \lambda \int_0^{ct} [Y(x)]^2 \mathrm{d}x\right\} \\ &\quad + \frac{1}{2} EIT^2 c \left\{\alpha^2 \left[Y'(ct)\right]^2 - \left[Y_1''(ct)\right]^2 - \lambda [Y(ct)]^2\right\} = 0 \end{aligned} \tag{3.31}$$

By use of Eqs. (3.19) and (3.30), from Eq. (3.31) we obtain the supplementary restraint condition at the compression wave front:

$$Y''(ct) = 0 \tag{3.32}$$

The governing equation (3.13, second term), the boundary conditions (3.18) and (3.19) and the supplementary restraint equation (3.32) compose the necessary and sufficient conditions for determining the buckling modes, the critical load parameter α^2 and the dynamic characteristic parameter $\lambda = \omega^2$.

4. Governing equations for dynamic buckling at the second stage

At the instant t of the second stage of the compression wave propagation, the traveling distance of the reflected wave front from the reflection end is $L_1 = ct$, as shown in Fig. 1b. The amplitude of axial force in the column is equal to $2N$ for the region $0 < x < L_1$ and N for the region $L_1 < x \leqslant L$.

For the dynamic buckling of the column at this stage, the buckling deflection $w(x, t)$ is written as

$$w(x, t) = w_1(x, t) \quad \text{for} \quad 0 \leqslant x \leqslant ct \tag{4.1}$$

$$w(x, t) = w_2(x, t) \quad \text{for} \quad ct \leqslant x \leqslant L \tag{4.2}$$

For brevity, we use the critical condition (3.28) instead of the Hamilton's theorem for the derivation of governing equations, boundary conditions and continuity conditions. The compressive deformation energy released in the instant course of the dynamic buckling is written as

$$U_{\text{pre}} = \frac{1}{2}N_1\int_0^{ct}(w_{1,x})^2\,\mathrm{d}x + \frac{1}{2}N_2\int_{ct}^{L}(w_{2,x})^2\,\mathrm{d}x \tag{4.3}$$

where

$$N_1 = 2N, \quad N_2 = N$$

The buckling deformation energy is

$$U_{\text{buc}} = \frac{1}{2}EI\int_0^{ct} w_{1,xx}^2\,\mathrm{d}x + \frac{1}{2}EI\int_{ct}^{L} w_{2,xx}^2\,\mathrm{d}x \tag{4.4}$$

As mentioned in Section 3.2, the quantity $\dot{u}_1^2$ is negligible as compared with $\dot{w}^2$. The buckling kinetic energy is written as

$$\Delta K = \frac{1}{2}\rho A\int_0^{ct}\dot{w}_1^2\,\mathrm{d}x + \frac{1}{2}\rho A\int_{ct}^{L}\dot{w}_2^2\,\mathrm{d}x \tag{4.5}$$

We write the deflection function into the variable-separated form:

$$w_1(x,t) = T(t)Y_1(x) \tag{4.6}$$

$$w_2(x,t) = T(t)Y_2(x) \tag{4.7}$$

Introducing Eqs. (4.3)–(4.5) into the critical condition (3.28), by use of Eqs. (4.6) and (4.7), we derive the first term of Eq. (3.13) and the following equation:

$$\begin{aligned}\alpha_1^2\int_0^{ct}[Y_1'(x)]^2\,\mathrm{d}x + \alpha_2^2\int_{ct}^{L}[Y_2'(x)]^2\,\mathrm{d}x - \int_0^{ct}[Y_1''(x)]^2\,\mathrm{d}x - \int_{ct}^{L}[Y_2''(x)]^2\,\mathrm{d}x - \lambda\int_0^{ct}[Y_1(x)]^2\,\mathrm{d}x \\ - \lambda\int_{ct}^{L}[Y_2(x)]^2\,\mathrm{d}x = 0\end{aligned} \tag{4.8}$$

where

$$\alpha_1^2 = \frac{N_1}{EI}, \quad \alpha_2^2 = \frac{N_2}{EI} \tag{4.9}$$

Integrating the left side of Eq. (4.8) by parts, we obtain the following governing equations (4.10) and (4.11), and the continuity conditions (4.12) and (4.13) at the reflected wave front:

$$\frac{\mathrm{d}^4Y_1}{\mathrm{d}x^4} + \alpha_1^2\frac{\mathrm{d}^2Y_1}{\mathrm{d}x^2} + \lambda Y_1 = 0, \quad \text{for} \quad 0 \leqslant x \leqslant ct \tag{4.10}$$

$$\frac{\mathrm{d}^4Y_2}{\mathrm{d}x^4} + \alpha_2^2\frac{\mathrm{d}^2Y_2}{\mathrm{d}x^2} + \lambda Y_2 = 0, \quad \text{for} \quad ct \leqslant x \leqslant L \tag{4.11}$$

$$Y_1'(ct) = Y_2'(ct) \tag{4.12}$$

$$\begin{aligned}&Y_1(ct) = Y_2(ct), \quad Y_1''(ct) = Y_2''(ct), \\ &\alpha_1^2Y_1'(ct) + Y_1'''(ct) = \alpha_2^2Y_2'(ct) + Y_2'''(ct)\end{aligned} \tag{4.13}$$

The following boundary conditions at both ends of the column are also derived from the partial integration of Eq. (4.8).

The boundary conditions at the reflection end $x = 0$ are written as

$$\begin{aligned} &Y_1(0) = 0 \quad \text{or} \quad \alpha_1^2 Y_1'(0) + Y_1'''(0) = 0 \\ &Y_1'(0) = 0 \quad \text{or} \quad Y_1''(0) = 0 \end{aligned} \tag{4.14}$$

For the loaded end $x = L$, we have the boundary conditions

$$\begin{aligned} &Y_1(L) = 0 \quad \text{or} \quad \alpha_1^2 Y_1'(L) + Y_1'''(L) = 0 \\ &Y_1'(L) = 0 \quad \text{or} \quad Y_1''(L) = 0 \end{aligned} \tag{4.15}$$

Eqs. (4.10)–(4.15) are insufficient for the determination of the problem solution. We can derive the supplementary equation by use of the critical condition (3.29). Introducing Eqs. (4.3)–(4.5) into Eq. (3.29), by use of Eq. (4.8) we derive the following equation.

$$\begin{aligned} \dot{U}_{\text{pre}} - \dot{U}_{\text{buc}} - \Delta\dot{K} = \frac{1}{2} EIT^2 c \Big\{ &\alpha_1^2 \big[Y_1'(ct)\big]^2 - \alpha_2^2 \big[Y_2'(ct)\big]^2 - \big[Y_1''(ct)\big]^2 \\ &+ \big[Y_2''(ct)\big]^2 - \lambda [Y_1(ct)]^2 + \lambda [Y_2(ct)]^2 \Big\} = 0 \end{aligned} \tag{4.16}$$

From Eq. (4.16), by use of Eqs. (4.12) and first and second terms of Eqs. (4.13) we obtain the following restraint equations at the reflected wave front:

$$Y_1'(ct) = 0 \quad Y_2'(ct) = 0 \tag{4.17}$$

The governing equations (4.10) and (4.11), the continuity condition (4.13), the boundary conditions (4.14) and (4.15), and the supplementary restraint equation (4.17) compose the necessary and sufficient conditions for determining the solution of the dynamic buckling problem in this section.

5. Solutions of equations

5.1. Solution of the equations for dynamic buckling at the first stage

For the convenience of analysis, we rewrite the governing equation (3.13, second term), the boundary condition (3.19) and the supplementary restraint equation (3.32) as follows:

$$Y''''(x) + \alpha^2 Y''(x) + \omega^2 Y(x) = 0 \quad (0 \leqslant x \leqslant L_1 = ct) \tag{5.1}$$

$$Y_1(ct) = 0, \quad Y_1'(ct) = 0 \tag{5.2}$$

$$Y_1''(ct) = 0 \tag{5.3}$$

For the boundary conditions of the loaded end list in Eqs. (3.18), we consider the two cases as follows:

(1) The column is simply supported at the loaded end.

$$Y(0) = 0, \quad Y''(0) = 0 \tag{5.4}$$

(2) The loaded end is clamped, but is movable in the x-axis direction.

$$Y(0) = 0, \quad Y'(0) = 0 \tag{5.5}$$

For the values of the parameters α^2 and ω, there are three kinds of cases: (a) $\alpha_1^2 > 2\omega$, (b) $\alpha_1^2 = 2\omega$, and (c) $\alpha_1^2 < 2\omega$. By the derivation, it is found that only if $\alpha_1^2 > 2\omega$, the governing equation (5.1) has the solution satisfying the restraint conditions (5.2), (5.3) and (5.4) or (5.5). The expression of the solution is written as follows.

$$Y(x) = D_1 \cos(\beta_1 x) + D_2 \sin(\beta_1 x) + D_3 \cos(\beta_2 x) + D_4 \sin(\beta_2 x) \tag{5.6}$$

where

$$\beta_1 = \sqrt{\frac{1}{2}\left[\alpha^2 + \sqrt{\alpha^4 - 4\omega^2}\right]}, \qquad \beta_2 = \sqrt{\frac{1}{2}\left[\alpha^2 - \sqrt{\alpha^4 - 4\omega^2}\right]} \tag{5.7}$$

$$\alpha^2 = \beta_1^2 + \beta_2^2, \quad \omega = \beta_1\beta_2 \tag{5.8}$$

For the column that is simply supported at the loaded end, introducing the expression (5.6) into Eqs. (5.2)–(5.4), we obtain a series of the values of the dimensionless parameters $\beta_1 L_1$ and $\beta_2 L_1$:

$$\beta_1 L_1 = (n+1)\pi, \quad \beta_2 L_1 = n\pi, \qquad n = 1, 2, 3, \ldots \tag{5.9}$$

For the column of which the loaded end is clamped but movable in the axial direction, introducing the expression (5.6) into Eqs. (5.2), (5.3) and (5.5), we obtain

$$\beta_1 L_1 = (n+2)\pi, \quad \beta_2 L_1 = n\pi, \qquad n = 1, 2, 3, \ldots \tag{5.10}$$

In the above-mentioned derivation, the integration constants D_i $(i = 1, 2, 3, 4)$ are also calculated except for an undetermined multiplying factor.

5.2. Solution of the equations for dynamic buckling at the second stage

5.2.1. Solution of Eqs. (4.10) and (4.11)

For the dynamic buckling of the column at the second stage of compression wave propagation, the conditions of determining solution consist of the governing equations (4.10) and (4.11), the continuity condition (4.13), the supplementary restraint equation (4.17), and the boundary conditions (4.14) and (4.15). Eqs. (4.13) and (4.17) are re-written as follows.

$$Y_1(ct) = Y_2(ct), \quad Y_1''(ct) = Y_2''(ct), \quad Y_1'''(ct) = Y_2'''(ct) \tag{5.11}$$

$$Y_1'(ct) = 0, \quad Y_2'(ct) = 0 \tag{5.12}$$

In order to compare the dynamic buckling of the column at this stage with that at the first stage, we consider the case where the loaded end is clamped but movable in the axial direction.

$$Y_1(L) = 0, \quad Y_1'(L) = 0 \tag{5.13}$$

For the boundary condition (4.14) at the reflection end, we consider the following two cases.
(1) The column is hinged at the reflection end.

$$Y(0) = 0, \quad Y''(0) = 0 \tag{5.14}$$

(2) The column is clamped at the reflection end.

$$Y(0) = 0, \quad Y'(0) = 0 \tag{5.15}$$

By derivation, it is found that only if $\alpha_1^2 > 2\omega > \alpha_2^2$, Eqs. (4.10) and (4.11) have the solution satisfying the restraint conditions (5.11), (5.12), (5.13) and (5.14) or (5.15). The expressions of the solution are written as

$$Y_1(x) = D_1\cos(\beta_1 x) + D_2\sin(\beta_1 x) + D_3\cos(\beta_2 x) + D_4\sin(\beta_2 x) \quad (0 \leqslant x \leqslant ct) \tag{5.16}$$

$$Y_2(x) = ch(\xi_1 x)[d_1\cos(\xi_2 x) + d_2\sin(\xi_2 x)] + sh(\xi_1 x)[d_3\cos(\xi_2 x) + d_4\sin(\xi_2 x)] \quad (ct \leqslant x \leqslant L) \tag{5.17}$$

where

$$\beta_1 = \sqrt{\frac{1}{2}\left[\alpha_1^2 + \sqrt{\alpha_1^4 - 4\omega^2}\right]}, \qquad \beta_2 = \sqrt{\frac{1}{2}\left[\alpha_1^2 - \sqrt{\alpha_1^4 - 4\omega^2}\right]} \tag{5.18}$$

$$\alpha_1^2 = \beta_1^2 + \beta_2^2, \quad \alpha_2^2 = \frac{1}{2}\alpha_1^2, \quad \omega = \beta_1\beta_2 \tag{5.19}$$

$$\xi_1 = \frac{1}{2}\sqrt{2\omega - \alpha_2^2}, \quad \xi_2 = \frac{1}{2}\sqrt{2\omega + \alpha_2^2} \tag{5.20}$$

The solution (5.16) represents the dominant buckling deformation in the region $0 \leqslant x \leqslant ct$ where the reflected wave is superposed on the initial compression wave, and the solution (5.17) represents the disturbance deformation in the other portion of the column.

5.2.2. The solution of the characteristic parameters

Introducing the expression (5.16) and (5.17) into Eqs. (5.11), (5.12), (5.13) and (5.14) or (5.15), we obtain nine linear algebraic equations for the eight constants D_i and d_i $(i = 1, 2, 3, 4)$. In accordance with the conditions under which the nontrivial solution exists for these equations, we derive the two algebraic characteristic-equation (5.21) for the characteristic parameters $\kappa_1 = \beta_1 L_1$ and $\kappa_2 = \beta_2 L_1$.

$$F_1(\kappa_1, \kappa_2, \eta) = 0, \quad F_2(\kappa_1, \kappa_2, \eta) = 0 \tag{5.21}$$

In Eq. (5.12), $\eta = (L - c\tau)/L$. After the roots of Eq. (5.21) is solved, the values of the parameters α_1^2 and $\omega = \sqrt{\lambda}$ are calculated by use of first and third terms of Eq. (5.19), and the corresponding buckling modes are computed by use of Eqs. (5.16) and (5.17).

6. Numerical results

6.1. Results for the dynamic buckling at the first stage of compression wave propagation

Let τ denote the critical buckling time and L_1 denote the traveling distance of the compression wave when the dynamic buckling occurs.

$$L_1 = c\tau, \quad L_2 = L - L_1 \tag{6.1}$$

When dynamic buckling deformation occurs in the portion $0 < x < c\tau$ of the column at this stage, the portion of the column before the wave front remains undisturbed. For the two types of boundary

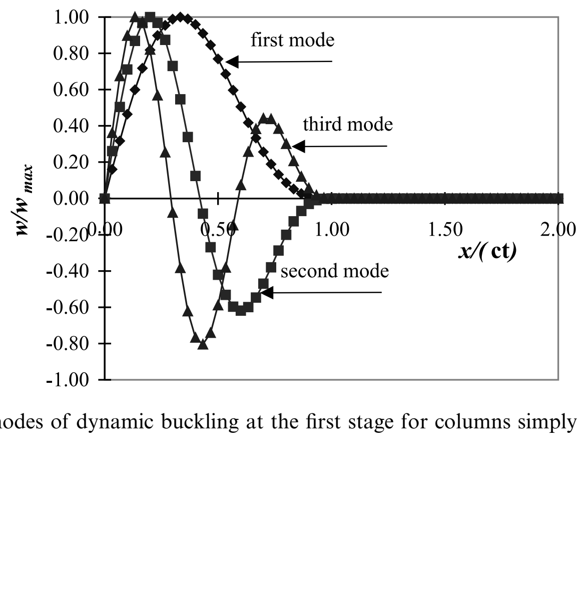

Fig. 2. The first three modes of dynamic buckling at the first stage for columns simply supported at loaded end.

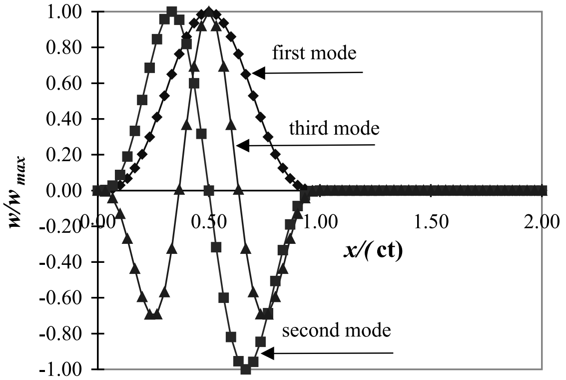

Fig. 3. The first three modes of dynamic buckling at the first stage for columns of which loaded end is clamped but movable in the axial direction.

Table 1
The values of $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ for dynamic buckling before compression wave reflection

	Loaded end is simply supported			Loaded end is clamped but axially movable		
	$n=1$	$n=2$	$n=3$	$n=1$	$n=2$	$n=3$
$\Lambda_1^{(n)}$	$5\pi^2$	$13\pi^2$	$25\pi^2$	$10\pi^2$	$20\pi^2$	$34\pi^2$
$\Lambda_2^{(n)}$	$2\pi^2$	$6\pi^2$	$12\pi^2$	$3\pi^2$	$8\pi^2$	$15\pi^2$

conditions considered in Section 5.1, the first three buckling modes are shown in Figs. 2 and 3. Fig. 2 corresponds to the column of which the loaded end is simply supported, and Fig. 3 corresponds to the column of which the loaded end is clamped, but movable in the axial direction.

We introduce the dimensionless critical-load parameter $\Lambda_1^{(n)}$ and the dimensionless dynamic-characteristic parameter $\Lambda_2^{(n)}$.

$$\Lambda_1^{(n)} = (\alpha_1 L_1)^2, \quad \Lambda_2^{(n)} = \omega L_1^2 \tag{6.2}$$

where the superscript n denotes that the value of the parameter is corresponds to the nth buckling mode. The relation of the critical force with the buckling time τ is expressed as

$$N_{cr}^{(n)} = \frac{\Lambda_1^{(n)} EI}{L_1^2} = \frac{\Lambda_1^{(n)} EI}{c^2 \tau^2} \tag{6.3}$$

Corresponding to the first three modes, the values of the parameters $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ are list in Table 1.

From Table 1, it can be seen that the values of the critical-load parameter $\Lambda_1^{(1)}$ are $5\pi^2$ and $10\pi^2$ respectively for the two types of loaded-end boundary conditions. The value of the corresponding critical-load parameter is $2.046\pi^2$ for the static buckling of the column that is simply supported at one end and clamped at another end. The value of the critical-load parameter is $4\pi^2$ for the static buckling of the column of which both ends are clamped but one end is movable in the axial direction. It is obvious that the critical load for the dynamic buckling is much higher than that for the static buckling.

From Eqs. (3.21) and (3.22), it can be seen that the buckling kinetic-energy ΔK in the right side of Eq. (3.28) is positively definite. It is that the buckling kinetic-energy term ΔK makes the critical force for the dynamic buckling much higher than that for the static buckling of the column. For the same reason, it can be concluded that omitting the term ΔK_2 in the expression of the buckling kinetic-energy ΔK, that is omission of the axial inertia effect, will make the obtained critical force be a little lower than its true value.

Assuming that the dynamic buckling takes place when the compression wave front arrives at the fixed end, the comparison of the first two dynamic modes with the static buckling modes is shown in Figs. 4 and 5. From Figs. 4 and 5 it is obvious that the dynamic modes are different from the static modes.

6.2. Comparison of theoretical results with experimental results

In order to compare the theoretical results in Section 6.1 with experimental results given by Tang and Zhu (1994), we introduce the following parameter.

$$\varepsilon = \frac{N_{cr}^{(1)}}{EA}, \quad \eta = \frac{c\tau}{r}, \quad r^2 = \frac{I}{A} \tag{6.4}$$

In Eq. (6.4), A denotes the cross-section area of the column. For the case $n = 1$, the formula (6.3) is rewritten as

$$\varepsilon\eta^2 = \Lambda_1^{(1)} \tag{6.5}$$

Introducing the value of the parameter $\Lambda_1^{(1)}$ from Table 1, the formula (6.5) for the column clamped at loaded end is written as

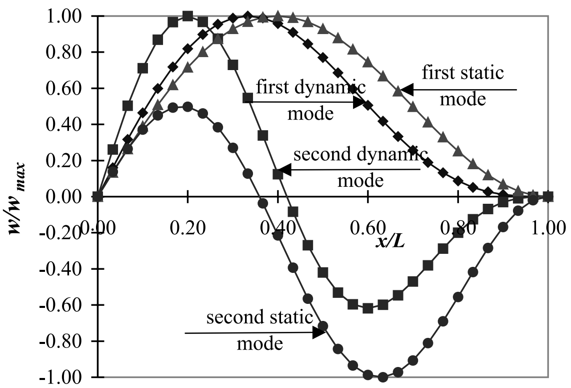

Fig. 4. Comparison of dynamic modes with static modes for columns that is simply supported at loaded end and clamped at other end.

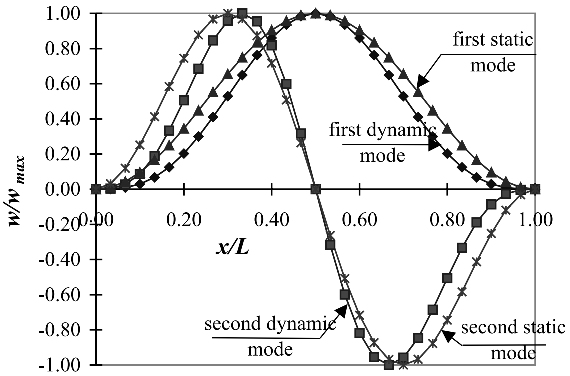

Fig. 5. Comparison of dynamic modes with static modes for columns of which loaded end is clamped but movable in the axial direction and the other end is clamped.

$$\varepsilon\eta^2 = 10\pi^2 \tag{6.6}$$

The experimental investigation of Tang and Zhu (1994) was confined to the dynamic buckling that occurs before compression wave is reflected from the fixed end. Only clamped boundary condition at the impact end was considered and only situation where the bar buckles in the first mode was dealt with. The experiment was carried out using a Hopkinson bar to detect critical buckling times of a specimen. The specimen is a long steel bar with rectangular cross-section $7.3 \times 10\ \text{mm}^2$. The specimen was impacted with a shorter steel-bar projectile at velocities from 6 to 22 m/s. Three pairs of strain gauges were placed on opposite sides of the specimen at three different positions. Before bending signals reached a measuring station, strains recorded by the pair of gauges would be identical. When recorded strain signals began to split into two different traces, dynamic buckling occurred. The time of strain record split in the first pair of opposite gauges was defined as the critical buckling time. For every impact velocity, an axial strain ε and a corresponding critical time τ was obtained. In this way, a series of experimental points in $\varepsilon - \eta$ plane was obtained. By a least square fitting, Tang and Zhu (1994) gave the following empirical formula.

$$\varepsilon_{\text{exp}}\eta_{\text{exp}}^2 = 9.4\pi^2 \tag{6.7}$$

It can be seen that the theoretical result predicted by the formula (6.6) is in reasonable agreement with the experiment result of Eq. (6.7). The experimental value at the right side of Eq. (6.7) is lower than the theoretical value of the formula (6.6). This situation may be owing to that the impact end of specimens was not completely clamped.

6.3. Results for the dynamic buckling at the second stage of compression wave propagation

At this stage, the first term of Eq. (6.1) denotes the propagation distance of the reflected wave front from the reflection end A when the dynamic buckling occurs, as shown in Fig. 1b. The dynamic buckling is caused mainly by the axial superposition force in the region $0 \leqslant x < L_1$. The numerical results are obtained by use of the solution in Section 5.2. In order to compare the results for the dynamic buckling at this stage with the results of Section 6.1, we consider the following two types of boundary conditions: (1) The reflection end is hinged; (2) The reflection end is clamped. For both cases, the loaded end is clamped but movable in the axial direction.

For the above-mentioned two types of the boundary conditions, the first three dynamic buckling modes are illustrated in Figs. 6–9, where the point $x/c\tau = 1$ at x-axis corresponds to the reflected wave front. From Figs. 6–9, it can be seen that the disturbance deformation in the region $c\tau < x < L$ decreases rapidly with

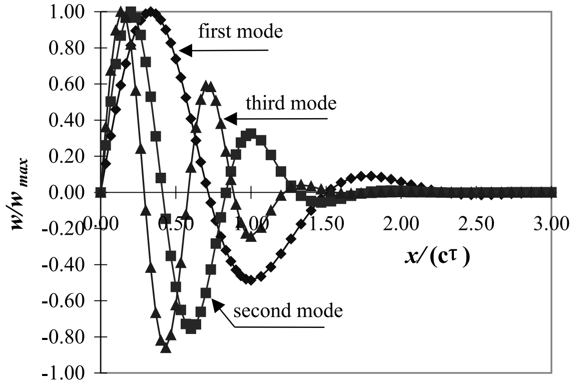

Fig. 6. The first three modes of dynamic buckling at the second stage for columns hinged at reflection end, $L/L_1 = 3$.

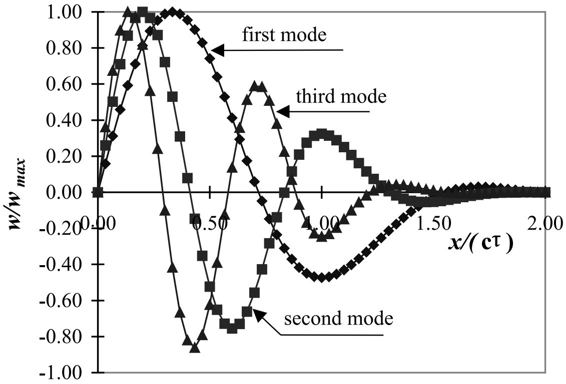

Fig. 7. The first three modes of dynamic buckling at the second stage for columns hinged at the reflection end, $L/L_1 = 2$.

the increment of the distance from the reflected wave front. The values of the parameters $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ corresponding to the different values of $k = L_2/L_1$ are list in Tables 2 and 3. At this stage, the value of the parameters $\Lambda_1^{(1)}$ is lower than the values $5\pi^2$ for the column of which the reflection end is hinged, and is

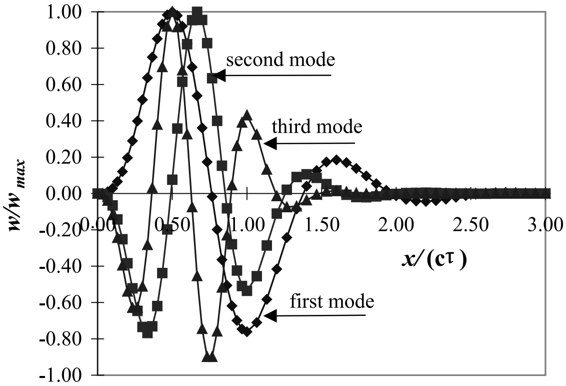

Fig. 8. The first three modes of dynamic buckling at the second stage for columns clamped at reflection end, $L/L_1 = 3$.

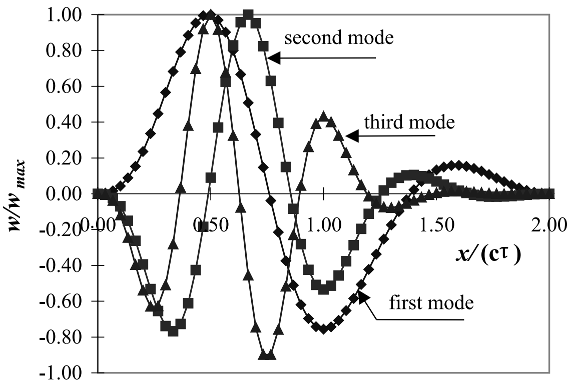

Fig. 9. The first three modes of dynamic buckling at the second stage for columns clamped at reflection end, $L/L_1 = 2$.

Table 2
The values of $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ for dynamic buckling at the second stage, for the column of which the reflection end is hinged

L_2/L_1		2.0	1.0	0.5	0.3	0.2	0.05
$\Lambda_1^{(n)}(\pi^2)$	$n = 1$	4.7442	4.7186	4.2697	4.0229	4.2163	4.7609
	$n = 2$	12.817	12.816	12.814	12.061	11.443	12.388
	$n = 3$	24.858	24.858	24.807	24.825	23.535	23.850
$\Lambda_2^{(n)}(\pi^2)$	$n = 1$	2.1272	2.1097	1.8652	1.6408	1.6957	1.9045
	$n = 2$	6.0915	6.0910	6.0831	5.6943	5.3201	5.7180
	$n = 3$	12.071	12.071	12.045	12.049	11.386	11.449

Table 3
The values of $\Lambda_1^{(n)}$ and $\Lambda_2^{(n)}$ for dynamic buckling at the second stage, for the column of which the reflection end is clamped

L_2/L_1		2.0	1.0	0.5	0.3	0.2	0.05
$\Lambda_1^{(n)}(\pi^2)$	$n = 1$	9.0911	8.6878	8.0038	6.4894	7.1854	9.0780
	$n = 2$	19.354	19.223	18.563	16.904	15.036	18.172
	$n = 3$	33.475	33.452	32.967	33.335	28.896	30.930
$\Lambda_2^{(n)}(\pi^2)$	$n = 1$	3.4205	3.2629	2.9588	2.0032	2.1663	2.7234
	$n = 2$	8.3234	8.2672	7.9611	7.1935	6.0884	7.2690
	$n = 3$	15.253	15.243	15.020	15.167	13.032	13.646

lower than the value $10\pi^2$ for the column of which the reflection end is clamped. For the dynamic buckling at this stage, $N_{cr}^{(n)}$ in Eq. (6.3) represents the critical value of the axial force N_1 in the region $0 < x < c\tau$.

7. Conclusion

(1) Two critical conditions for the dynamic buckling of columns have been derived on the basis of the consideration of energy transformation and conservation. The necessary and sufficient conditions for the solution of the dynamic buckling of the column are derived by use of the two critical conditions or by use of the adjacent-equilibrium criterion and the second critical condition.

A couple of characteristic equations for the two characteristic parameters are derived. The dynamic buckling modes, the critical load parameter and the dynamic characteristic parameter are calculated accurately from the solutions of the characteristic equations. Thus, the method of the characteristic-value analysis is presented for the elastic dynamic buckling of the column under an axial compression wave Simple formulas for the relation of the critical force with the critical buckling time are given.

(2) Transverse inertia effect has been determined quantitatively in terms of the value of the dynamic characteristic parameter. It is the transverse inertial effect that makes the critical force of the dynamic buckling be much higher than that of the static buckling for the column. Neglecting the axial inertia effect and the rotational inertia effect in the analysis will make the calculated value of critical force be a little lower than its true value.

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